Asymptotic behaviour of the simple random walk on the 2-dimensional comb *

Daniela Bertacchi
Università di Milano-Bicocca
Dipartimento di Matematica e Applicazioni
Via Cozzi 53, 20125 Milano, Italy
daniela.bertacchi@unimib.it

Abstract

We analyze the differences between the horizontal and the vertical component of the simple random walk on the 2-dimensional comb. In particular we evaluate by combinatorial methods the asymptotic behaviour of the expected value of the distance from the origin, the maximal deviation and the maximal span in n steps, proving that for all these quantities the order is $n^{1/4}$ for the horizontal projection and $n^{1/2}$ for the vertical one (the exact constants are determined). Then we rescale the two projections of the random walk dividing by $n^{1/4}$ and $n^{1/2}$ the horizontal and vertical ones, respectively. The limit process is obtained. As a corollary of the estimate of the expected value of the maximal deviation, the walk dimension is determined, showing that the Einstein relation between the fractal, spectral and walk dimensions does not hold on the comb.

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1 Introduction and main results

The 2-dimensional comb $\mathbf{C}_2$ is maybe the simplest example of inhomogeneous graph. It is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the x-axis (see Figure 1). Many features of the simple random walk on this graph has been matter of former investigations. Local limit theorems were first obtained by Weiss and Havlin [19] and then extended to higher dimensions by Gerl [10] and Cassi and Regina [4]. More recently, Krishnapur and Peres [14] have shown that on $\mathbf{C}_2$ two independent walkers meet only finitely many times almost surely. This result, together with the space-time asymptotic estimates obtained in [3] for the n-step transition probabilities, suggests that the walker spends most of the time on some tooth of the comb, that is moving along the vertical direction. Indeed in [3, Section 10], it has been remarked that, if k/n goes to zero with a certain speed, then $p^{(2n)}\big((2k,0),(0,0)\big)/p^{(2n)}\big((0,2k),(0,0)\big) \stackrel{n\to\infty}{\longrightarrow} 0$.

Moreover the results in [3] imply that there are no sub-Gaussian estimate of the transition probabilities on $\mathbf{C}_2$. Such estimates have been found on many graphs: by Jones [12] on the

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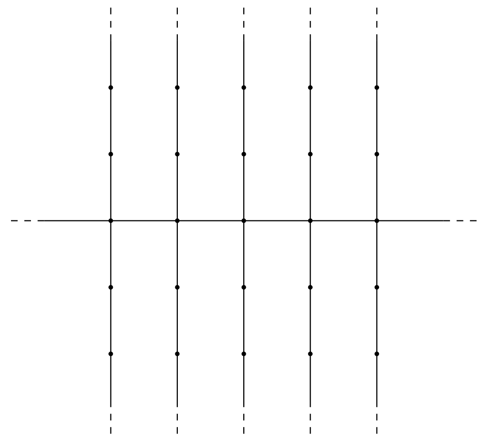

Figure 1: The 2-dimensional comb.

2-dimensional Sierpiński graph, by Barlow and Bass [1] on the graphical Sierpiński carpet and on rather general graphs by Grigor'yan and Telcs ([11, 18]). These estimates involve three exponents which are usually associated to infinite graphs: the *spectral dimension* δ_s (which is by definition twice the exponent of n^{-1} in local limit theorems), the *fractal dimension* δ_f (which is the growth exponent) and the *walk dimension* δ_w. These dimensions are in typical cases linked by the so-called *Einstein relation*: $\delta_s \delta_w = 2\delta_f$ (see Telcs [16, 17]). The first two dimensions are known for $\mathbf{C}_2$: $\delta_s = 3/2$ and $\delta_f = 2$. In this paper we compute $\delta_w = 2$, thus showing that the relation does not hold for this graph.

In order to point out the different behaviour of the random walk along the two directions we analyze the asymptotic behaviour of the expected value of the distance from the origin reached by the walker in n steps. Different concepts of distance are considered: position after n steps, maximal deviation and maximal span. Section 2 is devoted to these definitions, and to the necessary preliminaries, such as the definitions of random walk on a graph and of generating function. The expression of the generating function of the transition probabilities of the simple random walk on $\mathbf{C}_2$ is recalled.

In Sections 3, 4 and 5 we prove the asymptotic estimates of the expected value of the distance of the walk from the origin after n steps, of its maximal deviation from the origin and of its maximal span respectively. The proofs are based on a Darboux type transfer theorem: we refer to [9, Corollary 2], but one may also refer to [2] and to the Hardy-Littlewood-Karamata theorem (see for instance [8]). This theorem (as far as we are concerned) claims that if

$$F(z) := \sum_{n=0}^{\infty} a_n z^n \stackrel{z \to 1^-}{\sim} \frac{C}{(1-z)^{\alpha}}, \qquad \alpha \notin \{0, -1, -2, \dots\},$$

and $F(z)$ is analytic in some domain, with the exception of $z = 1$, then

$$a_n \stackrel{n \to \infty}{\sim} \frac{C}{\Gamma(\alpha)} n^{\alpha - 1}.$$

The aim of our computation is then to determine an explicit expression of the generating functions of the sequence of expected values of the random variables we are interested in.

This is done employing the combinatorial methods used by Panny and Prodinger in [15]. We also refer to that paper for a comparison between our results and the analogous results for the simple random walk on $\mathbb{Z}$. Theorems 3.1, 3.2, 4.4, 4.10, 5.1 and 5.2 prove that the expected values of the distances along the horizontal direction are all of order $n^{1/4}$, and along the vertical direction they are of order $n^{1/2}$ (the exact constants are determined).

Since the notions of maximal deviation in n steps and of first exit time from a k-ball are closely related, we determine the walk dimension of $\mathbf{C}_2$ in Section 4.

In Section 6 we deal with the limit of the process obtained dividing by $n^{1/4}$ and by $n^{1/2}$ respectively the continuous time interpolation of the horizontal and vertical projections of the position after n steps. As one would expect the limit of the vertical component is the Brownian motion, while the limit of the horizontal component is less obvious (it is a Brownian motion indexed by the local time at 0 of the vertical component). This scaling limit is determined in Theorem 6.1.

Finally, Section 7 is devoted to a discussion of the results, remarks and open questions.

2 Preliminaries

The simple random walk on a graph is a sequence of random variables $\{S_n\}_{n\geq 0}$, where S_n represent the position of the walker at time n, such that if x and y are vertices which are neighbours then

$$p(x,y) := \mathbb{P}\left(S_{n+1}=y|S_n=x\right) = \frac{1}{\deg(x)},$$

where $\deg(x)$ is the number of neighbours of x, otherwise $p(x,y)=0$. In particular on $\mathbf{C}_2$ the non-zero transition probabilities $p(x,y)$ are equal to $1/4$ if x is on the horizontal axis, and they are equal to $1/2$ otherwise.

Given $x,y\in\mathbf{C}_2$, let

$$p^{(n)}(x,y) := \mathbb{P}\left(S_n=y|S_0=x\right), \qquad n\geq 0,$$

be the n-step transition probability from x to y. Recall that the generating function of a sequence $\{a_n\}_{n\geq 0}$ is the power series $\sum_{n\geq 0} a_n z^n$; by definition the Green function associated to the random walk on a graph X is the family of generating functions of the sequences $\{p^{(n)}(x,y)\}_{n\geq 0}$, $x,y\in X$, that is

$$G(x,y|z) = \sum_{n\geq 0} p^{(n)}(x,y) z^n.$$

The Green function, with $x=(0,0)$, on $\mathbf{C}_2$ can be written explicitly as (see [3])

$$G\big((0,0),(k,l)|z\big) = \begin{cases} \frac{1}{2}\, G(z)(F_1(z))^{|k|}(F_2(z))^{|l|}, & \text{if } l\neq 0,\\ G(z)(F_1(z))^{|k|}, & \text{if } l=0,\end{cases}$$

where

$$G(z) = \frac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}};$$
$$F_1(z) = \frac{1+\sqrt{1-z^2}-\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}}{z};$$
$$F_2(z) = \frac{1-\sqrt{1-z^2}}{z}.$$

We refer to [21, Section 1.1] for more details on the random walks on graphs, transition probabilities and generating functions.

When we consider the walk up to time n, different concepts of "distance" arise. We consider two equivalent norms on $\mathbf{C}_2$: for a given vertex (x, y) define

$$\|(x,y)\|_1 = |x| + |y|, \qquad \|(x,y)\|_\infty = \max\{|x|, |y|\}.$$

Note that $\|\cdot\|_1$ is the usual distance on the graph. In the following section we deal not only with the asymptotic behaviour of $\mathbb{E}[|S_n^x|]$ and $\mathbb{E}[|S_n^y|]$ (we use the notation $S_n = (S_n^x, S_n^y)$), but also with the asymptotics of the expected value of other random variables, which represent the (horizontal and vertical) maximal deviation and the span of the walk.

Definition 2.1. *a. The maximal deviations in n steps are defined as*

$$D_n^x := \max\{|S_i^x| : 0 \le i \le n\};$$
$$D_n^y := \max\{|S_i^y| : 0 \le i \le n\}.$$

b. The maximal spans in n steps are defined as

$$M_n^x := \max\{S_i^x - S_j^x : 0 \le i, j \le n\},$$
$$M_n^y := \max\{S_i^y - S_j^y : 0 \le i, j \le n\}.$$

3 Mean distance

Theorem 3.1.

$$\mathbb{E}[|S_n^x|] \overset{n\to\infty}{\sim} \frac{1}{2^{3/4}\Gamma(5/4)} n^{1/4}.$$

Proof. Since for $k \neq 0$,

$$\mathbb{P}\left(|S_n^x| = k\right) = 2\sum_{l\in\mathbb{Z}} p^{(n)}\big((0,0),(k,l)\big),$$

it is clear that (exchanging the order of summation)

$$\sum_{n=0}^{\infty} \mathbb{E}[|S_n^x|] z^n = 2\sum_{k=1}^{\infty} k \sum_{l\in\mathbb{Z}} G\big((0,0),(k,l)|z\big).$$

By elementary computation one obtains

$$\sum_{n=0}^{\infty} \mathbb{E}[|S_n^x|] z^n = G(z)\,\frac{F_1(z)}{(1-F_1(z))^2}\,\frac{1}{1-F_2(z)}$$
$$\overset{z\to 1^-}{\sim} \frac{1}{2^{3/4}(1-z)^{5/4}}.$$

Thus, applying [9, Corollary 2] we obtain the thesis. □

Theorem 3.2.

$$\mathbb{E}[|S_n^y|] \overset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\, n^{1/2}.$$

Proof. As in the proof of the previous theorem, for $l \neq 0$, we write

$$\mathbb{P}\left(|S_n^y| = l\right) = 2\sum_{k\in\mathbb{Z}} p^{(n)}\big((0,0),(k,l)\big),$$

and

$$\sum_{n=0}^{\infty} \mathbb{E}[|S_n^y|] z^n = 2\sum_{l=1}^{\infty} l \sum_{k\in\mathbb{Z}} G\big((0,0),(k,l)|z\big),$$

Thus

$$\sum_{n=0}^{\infty} \mathbb{E}[|S_n^y|] z^n = G(z)\,\frac{1+F_1(z)}{1-F_1(z)}\,\frac{F_2(z)}{(1-F_2(z))^2} \overset{z\to 1^-}{\sim} \frac{1}{2^{1/2}(1-z)^{3/2}}.$$

Apply [9, Corollary 2], recalling that $\Gamma(3/2) = \sqrt{\pi}/2$, to conclude. □

Corollary 3.3. $\mathbb{E}[\|S_n\|_1]$ *and* $\mathbb{E}[\|S_n\|_\infty]$ *are both asymptotic, as* n *goes to infinity, to* $\sqrt{\frac{2}{\pi}}\, n^{1/2}$.

4 Mean maximal deviation and walk dimension

4.1 Maximal horizontal deviation

In order to compute the generating function of $\{\mathbb{E}[D_n^x]\}_{n\geq 0}$, we first need an expression for another generating function.

Lemma 4.1. *Let* $h \in \mathbb{N}\cup\{0\}$, $l \in \{0,\dots,h\}$. *The generating function of the sequence* $\{\mathbb{P}\left(D_n^x \leq h, S_n^x = l\right)\}_{n\geq 0}$ *is*

$$\psi_{h,l}\left(\frac{1-\sqrt{1-z^2}}{2z}\right)\cdot\frac{2(1-\sqrt{1-z^2})}{z(\sqrt{1-z^2}-1+z)},$$

where $\psi_{h,l}(z)$ *is the generating function of the number of paths on* $\mathbb{Z}$ *of length* n*, from* 0 *to* l *with maximal deviation less or equal to* h*.*

Proof. Note that the paths we are interested in have no bound on the vertical excursions. Thus we may decompose the walk into its horizontal and vertical components, and consider each horizontal step as a vertical excursion (whose length might be zero) coming back to the origin plus a step along the horizontal direction.

Keeping this decomposition in mind, it is clear that the generating function of the sequence $\{\mathbb{P}\left(D_n^x \leq h, S_n = (l,0)\right)\}_{n\geq 0}$ is

$$\psi_{h,l}\left(\frac{z\widetilde{G}(0,0|z)}{4}\right),$$

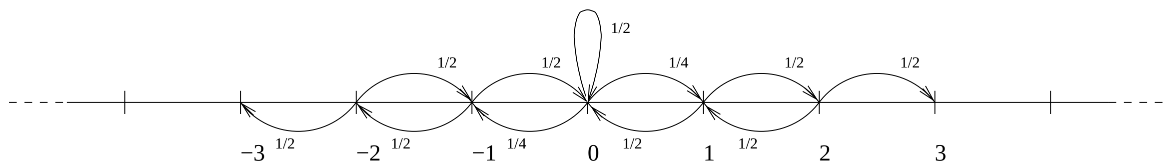

Figure 2: The vertical component of the simple random walk on $\mathbf{C}_2$.

where $\widetilde{G}(0,j|z)$ is the generating function of the probabilities of the n-step excursions along one single tooth of the comb (that is paths in Figure 2 which do not use the loop at zero), from $(x,0)$ to (x,j).

Moreover we must admit a final excursion from $(l,0)$ to (l,j) for some $j \in \mathbb{Z}$, that is we must multiply by

$$E(z) := \widetilde{G}(0,0|z) + 2\sum_{j=1}^{\infty} \widetilde{G}(0,j|z),$$

and the generating function of $\{\mathbb{P}\left(D_n^x \leq h, S_n^x = l\right)\}_{n\geq 0}$ is

$$\psi_{h,l}\left(\frac{z\widetilde{G}(0,0|z)}{4}\right) \cdot E(z).$$

Using [21, Lemma 1.13] and reversibility (see [21, Section 1.2.A]), it is not difficult to compute

$$\widetilde{G}(0,0|z) = \frac{2\left(1-\sqrt{1-z^2}\right)}{z^2},$$
$$\widetilde{G}(0,j|z) = \frac{1}{2}\,\widetilde{G}(0,0|z)\left(\frac{1-\sqrt{1-z^2}}{z}\right)^j, \quad j \neq 0.$$

The thesis is obtained noting that

$$\frac{z\widetilde{G}(0,0|z)}{4} = \frac{1-\sqrt{1-z^2}}{2z},$$
$$E(z) = \frac{2(1-\sqrt{1-z^2})}{z(\sqrt{1-z^2}-1+z)}.$$

□

Proposition 4.2.

$$\sum_{n=0}^{\infty} \mathbb{E}[D_n^x]z^n = 2\,\frac{1+6v^2+v^4}{(1-v)^4}\sum_{h\geq 1}\frac{v^h}{1+v^{2h}},$$

where v is such that

$$\frac{v}{1+v^2} = \frac{1-\sqrt{1-z^2}}{2z}. \tag{1}$$

Proof. Let $\psi_{h,l}(z)$ be as in Lemma 4.1, and put $\psi_h(z) = \sum_{|l|\le h} \psi_{h,l}(z)$. Then the generating function of $\{\mathbb{P}(D_n^x \le h)\}_{n\ge 0}$ is

$$H_h(z) := \psi_h\left(\frac{1-\sqrt{1-z^2}}{2z}\right)\cdot\frac{2\left(1-\sqrt{1-z^2}\right)}{z(\sqrt{1-z^2}-1+z)}.$$

Thus we may write the generating function of $\mathbb{E}[D_n^x]$ as

$$\sum_{n=0}^{\infty}\left(\sum_{h=0}^{\infty}\mathbb{P}(D_n^x > h)\right)z^n = \sum_{h=0}^{\infty}\left(\frac{1}{1-z}-H_h(z)\right). \tag{2}$$

An explicit expression for ψ_h has been determined by Panny and Prodinger in [15, Theorem 2.2]:

$$\psi_h(w) = \frac{(1+v^2)(1-v^{h+1})^2}{(1-v)^2(1+v^{2h+2})},$$

where $w = v/(1+v^2)$. By the definition of $H_h(z)$ it is clear that the relation between z and v is set by equation (1). Then it is just a matter of computation to obtain

$$\begin{aligned} H_h(z) &= \frac{(1+6v^2+v^4)(1-v^{h+1})^2}{(1-v)^4(1+v^{2(h+1)})}, \\ \frac{1}{1-z} &= \frac{1+6v^2+v^4}{(1-v)^4}, \end{aligned}$$

whence, substituting in (2), the thesis. □

Lemma 4.3.

$$\sum_{h\ge 1}\frac{v^h}{1+v^{2h}} \overset{v\to 1^-}{\sim} \frac{\pi}{4(1-v)}.$$

Proof. The proof is quite standard, we report it here for completeness. Put $v = e^{-t}$, $g(w) = e^{-w}/(1+e^{-2w})$ and consider

$$f(t) := \sum_{h\ge 1}\frac{e^{-ht}}{1+e^{-2ht}} = \sum_{h\ge 1} g(ht).$$

The Mellin transform of f is:

$$\begin{aligned} f^*(s) &= \int_0^{\infty}\sum_{h\ge 1} g(ht)\, t^{s-1}dt \\ &= \zeta(s)\int_0^{\infty}\sum_{\lambda\ge 0}(-1)^{\lambda}e^{-(2\lambda+1)w}w^{s-1}dw, \end{aligned}$$

where $\zeta(s) = \sum_{h\ge 1} h^{-s}$ is the Riemann zeta function. The knowledge of the behaviour of $f^*(s)$ in a neighbourhood of 1, will give us the behaviour of $f(t)$ in a neighbourhood of 0. Substitute $y = (2\lambda+1)w$ to obtain

$$f^*(s) = \zeta(s)\kappa(s)\Gamma(s),$$

where $\kappa(s) = \sum_{\lambda\geq 0} \frac{(-1)^\lambda}{(1+2\lambda)^s}$, and $\Gamma(s) = \int_0^\infty e^{-y}y^{s-1}dy$ is the gamma function. Since as $s \to 1^+$

$$\zeta(s) = \frac{1}{s-1} + O(1),$$
$$\Gamma(s) = 1 + O(s-1),$$

we are left with the computation of the asymptotic behaviour of $\kappa(s)$. We may write

$$\kappa(s) = \frac{1}{4^s}\left(\zeta(s, 1/4) - \zeta(s, 3/4)\right),$$

where $\zeta(s,a) = \sum_{h\geq 0}(a+h)^{-s}$ is the Hurwitz zeta function. Thus using the expansion of $\zeta(s,a)$ for s close to 1 (see [20, Formula 13.21]) we obtain

$$\kappa(s) = \frac{\pi}{4} + O(s-1).$$

Hence we get

$$f^*(s) \overset{s\to 1^+}{\sim} \frac{\pi}{4(s-1)}.$$

Applying [6, Theorem 1, p.115],

$$f(t) \overset{t\to 0^+}{\sim} \frac{\pi}{4t},$$

which, substituting $t = -\log(1-(1-v))$, gives the thesis. □

Theorem 4.4.

$$\mathbb{E}[D_n^x] \overset{n\to\infty}{\sim} \frac{2^{-7/4}\pi}{\Gamma(5/4)} n^{1/4}.$$

Proof. By Proposition 4.2 and Lemma 4.3, it is clear that

$$\sum_{n\geq 0} \mathbb{E}[D_n^x] z^n \overset{v\to 1^-}{\sim} \frac{4\pi}{(1-v)^5}. \tag{3}$$

Choosing the solution v of equation (1) which is smaller than 1 when z is smaller than 1, and substituting it in (3) we obtain

$$\sum_{n=0}^{\infty} \mathbb{E}[D_n^x] z^n \overset{z\to 1^-}{\sim} \frac{2^{-7/4}\pi}{(1-z)^{5/4}},$$

and (by [9, Corollary 2]) the thesis. □

4.2 Maximal vertical deviation

We recall a result which can be found in [15, Theorem 2.1], which is useful in the sequel.

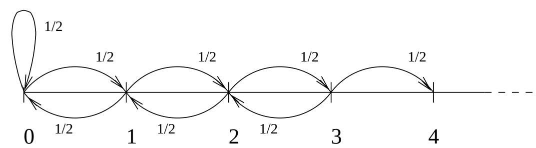

Figure 3: The absolute value of the vertical component of the simple random walk on $\mathbf{C}_2$.

Lemma 4.5. *Let $\mathbf{A}_i(z/2)$ be the $(i+1)\times(i+1)$ matrix*

$$\begin{bmatrix} 1 & -z/2 & 0 & \cdots & 0 \\ -z/2 & 1 & -z/2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -z/2 \\ 0 & \cdots & 0 & -z/2 & 1 \end{bmatrix}$$

and let $a_i(z/2)$ be its determinant. Then

$$a_i(z/2) = \frac{1-v^{2i+4}}{(1-v^2)(1+v^2)^{i+1}}, \tag{4}$$

where

$$\frac{v}{1+v^2} = \frac{z}{2}. \tag{5}$$

For the sake of simplicity, in the sequel we write $\mathbf{A}_i$ and a_i instead of $\mathbf{A}_i(z/2)$ and $a_i(z/2)$ respectively.

Lemma 4.6. *Let $h \in \mathbb{N}\cup\{0\}$, $l \in \{0,\dots,h\}$. The generating function of the sequence $\{\mathbb{P}\,(D^y_n \le h, S^y_n = l)\}_{n\ge 0}$ is*

$$\widehat{\psi}_{h,l}(z) = \frac{(z/2)^l a_{h-l-1}}{(1-z/2)a_{h-1} - z^2 a_{h-2}/4}. \tag{6}$$

Proof. Consider the absolute value of the vertical projection of the random walk as the random walk on the non negative integers with one-step transition probabilites described in Figure 3. Note that $\widehat{\psi}_{h,l}(z)$, $l = 0,\dots,h$, is determined by the linear system:

$$\left[\begin{array}{c|c} (1-z/2) & -z/2 \quad 0 \quad \cdots \quad 0 \\ \hline -z/2 & \\ 0 & \\ \vdots & \mathbf{A}_{h-1} \\ 0 & \end{array}\right] \begin{bmatrix} \widehat{\psi}_{h,0}(z) \\ \vdots \\ \vdots \\ \widehat{\psi}_{h,h}(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Using Cramer's rule (compare with [15, Theorem 2.1]), we obtain the thesis. $\square$

Lemma 4.7. *The generating function of $\{\mathbb{P}[D^y_n \le h]\}_{n\ge 0}$ is (written as a function of v)*

$$\widehat{\psi}_h(z) = \frac{(1+v^2)(1-v^{h+1})(1-v^{h+2})}{(1-v)^2(1+v^{2h+3})}.$$

Proof. From (6) and (4) we get

$$\widehat{\psi}_{h,l}(z) = \frac{v^l(1+v^2)(1-v^{2h-2l+2})}{(1-v)(1+v^{2h+3})},$$

whence, summing l from 0 to h, the proof is complete. □

Proposition 4.8, Lemma 4.9 and Theorem 4.10 are the analogs of Proposition 4.2, Lemma 4.3 and Theorem 4.4 respectively. Therefore, their proofs are omitted.

Proposition 4.8. *Let v be such that* (5) *holds. Then*

$$\sum_{n=0}^{\infty} \mathbb{E}[D_n^y] z^n = \frac{(1+v^2)(1+v)}{(1-v)^2} \sum_{h\geq 1} \frac{v^h}{(1+v^{2h+1})}. \tag{7}$$

Lemma 4.9.

$$\sum_{h\geq 1} \frac{v^h}{1+v^{2h+1}} \overset{v\to 1^-}{\sim} \frac{\pi}{4(1-v)}.$$

Theorem 4.10.

$$\mathbb{E}[D_n^y] \overset{n\to\infty}{\sim} \sqrt{\frac{\pi}{2}}\, n^{1/2}.$$

Corollary 4.11. *Both* $\mathbb{E}[\max_{0\leq i\leq n} \|S_i\|_1]$ *and* $\mathbb{E}[\max_{0\leq i\leq n} \|S_i\|_\infty]$*, as n goes to infinity, are asymptotic to* $\sqrt{\pi/2}\, n^{1/2}$.

Proof. Apply the results of Theorem 4.4 and Theorem 4.10 to the following inequalities

$$D_n^y \leq \max_{0\leq i\leq n} \|S_i\|_\infty \leq \max_{0\leq i\leq n} \|S_i\|_1 \leq D_n^x + D_n^y.$$

□

4.3 Walk dimension

The maximal deviation of the walk in n steps is linked to the first exit time from a ball of radius k. Indeed if we put $T_k = \min\{i : S_i \notin B_k\}$, where B_k is the ball of radius k centered in $(0,0)$, then

$$\left(\max_{0\leq i\leq n} \|S_i\| \leq k\right) = (T_k \geq n).$$

Clearly the radius of the ball and $\|S_i\|$ must be computed with respect to the same norm on the graph. We write T_k^∞ and T_k^1 for the exit times with respect to the two norms we defined in Section 2.

Recall now that given the simple random walk on a graph, if $\mathbb{E}[T_n]$ is of order n^α, then by definition α is the walk dimension of the graph. Usually, the norm with the respect to which the radius is computed is $\|\cdot\|_1$, but as we will show, we may equivalently consider $\|\cdot\|_\infty$. Therefore we are now interested in the asymptotic behaviour of $\mathbb{E}[T_n^\infty]$.

Proposition 4.12. $\mathbb{E}[T_n^\infty] \overset{n\to\infty}{\sim} n^2$.

Proof. We write

$$\mathbb{E}[T_n^\infty] = \sum_{k\geq 0} \mathbb{P}(T_n^\infty > k),$$

that is, $\mathbb{E}[T_n^\infty]$ is equal to $\Theta_n(1)$, where $\Theta_n(z)$ is the generating function of the sequence $\{\mathbb{P}\,(T_n^\infty > k)\}_{k\geq 0}$. Let us observe that $\mathbb{P}(T_n^\infty > k) = \mathbb{P}(\max_{0\leq i\leq k} \|S_i\|_\infty \leq n)$. We claim that

$$\Theta_n(z) = \frac{(1+w_n^2)(1-w_n^{n+1})^2}{(1-w_n)^2(1+w_n^{2n+2})} \cdot \frac{(1+v^2)(1-v^{n+2})(1-v^{n+1})}{(1-v)(1-v^{2n+4})} \tag{8}$$

where v is such that (5) holds and $w_n/(1+w_n^2) = v(1-v^{2n+2})/(2(1-v^{2n+4}))$. Then since for $v=1$ (choosing the solution w_n which is bounded in some neighbourhood of $v=0$)

$$w_n = 1 + \frac{1-\sqrt{2n+3}}{n+1},$$

we get that

$$\Theta_n(1) \overset{n\to\infty}{\sim} n^2.$$

We are left with the proof of equation (8). We proceed as in Lemma 4.1, that is we separately consider the two components of the walk. Let $\widetilde{G}_n(0,l|z)$ be the generating function of the probabilites of the m-step excursions along one single tooth, ending at a height l or $-l$, with maximal deviation bounded by n (see Figure 3). Put $\widetilde{G}_n(z) := \widetilde{G}_n(0,0|z)$. Then

$$\Theta_n(z) = \psi_n\left(\frac{z\widetilde{G}_n(z)}{4}\right) \sum_{l=0}^{n} \widetilde{G}_n(0,l|z),$$

where $\psi_n(z)$ is the generating function we already used in Proposition 4.2. By [21, Lemma 1.13]

$$\widetilde{G}_n(z) = \frac{1}{1 - z\widetilde{F}_n(1,0|z)/2},$$

where $\widetilde{F}_n(l,0|z)$ are the generating functions of the m-step "purely vertical" excursions, from (x,l) to $(x,0)$, with maximal deviation bounded by n. The $\widetilde{F}_n$'s are the solutions of the linear system:

$$\mathbf{A}_{n-1} \cdot \begin{bmatrix} \widetilde{F}_n(1,0|z) \\ \widetilde{F}_n(2,0|z) \\ \vdots \\ \widetilde{F}_n(n,0|z) \end{bmatrix} = \begin{bmatrix} z/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{9}$$

Then, with the substitution (5) (compare with [15]), we get

$$\widetilde{F}_n(l,0|z) = \frac{v^l(1-v^{2n-2l+2})}{(1-v^{2n+2})}, \qquad l = 1,\ldots,n,$$

$$\widetilde{G}_n(z) = \frac{(1+v^2)(1-v^{2n+2})}{(1-v^{2n+4})}.$$

Thus, $z\widetilde{G}_n(z)/4 = v(1-v^{2n+2})/(2(1-v^{2n+4})) = w_n/(1+w_n^2)$. For the final excursion we must compute $\widetilde{G}_n(0,l|z)$, for $l = 1,\dots n$. This can be done writing and solving n linear systems in the spirit of (9) to obtain

$$\widetilde{G}_n(0,l|z) = \frac{v^l(1+v^2)(1-v^{2n-2l+2})}{(1-v^{2n+4})},$$
$$\sum_{l=0}^{n} \widetilde{G}_n(0,l|z) = \frac{(1+v^2)(1-v^{n+1})(1-v^{n+2})}{(1-v)(1-v^{2n+4})}.$$

□

Corollary 4.13.

$$\mathbb{E}[T_n^1] \overset{n\to\infty}{\sim} n^2.$$

Proof. Define $T_{n,\alpha}$ as the first exit time from the rectangle $\{(x,y) : |x| \le \alpha h, |y| \le (1-\alpha)h\}$, and note that $T_{n,\alpha} \le T_n^1 \le T_n^\infty$. A calculation similar to that of Proposition 4.12, shows that $\mathbb{E}[T_{n,\alpha}] \overset{n\to\infty}{\sim} (1-\alpha)^2 n^2$. Since α is an arbitrary positive constant, we are done. □

Corollary 4.14. *The walk dimension of* $\mathbf{C}_2$ *is equal to* 2.

5 Mean maximal span

Theorem 5.1. $\mathbb{E}[M_n^x] \overset{n\to\infty}{\sim} \frac{2^{1/4}}{\Gamma(5/4)} n^{1/4}$.

Proof. Let $m_n^x = \max\{S_i^x : 0 \le i \le n\}$. Then it is clear that $\mathbb{E}[M_n^x] = 2\mathbb{E}[m_n^x]$. Our first aim is to compute the generating function of $\{\mathbb{P}(m_n^x \le h)\}_{n\ge 0}$, which we denote by $\widetilde{\Psi}_h(z)$. Then

$$\widetilde{\Psi}_h(z) = \lim_{k\to\infty} \sum_{l=-k}^{h} \widetilde{\Psi}_{h,k;l}(z) \cdot E(z),$$

where $\widetilde{\Psi}_{h,k;l}(z)$ is the generating function of the probabilities of the n-step paths such that $-k \le S_i^x \le h$ for $0 \le i \le n$, $S_n = (l,0)$ and $S_{n-1}^x \ne l$, while $E(z)$ was defined and computed in Lemma 4.1. Let us note that $\widetilde{\Psi}_{h,k;l}(z) = \Psi_{h,k;l}(z\widetilde{G}(0,0|z)/4)$, where $\Psi_{h,k;l}(z)$ is the generating function of the number of the n-step paths on $\mathbb{Z}$ which stay in the interval between $-k$ and h, and end at l. The functions $\Psi_{h,k;l}(w)$ are determined by the linear system used in [15, Theorem 4.1] to determine $\Psi_{h,k;0}(w)$. Thus

$$\Psi_{h,k;l}(w) = \frac{w^l a_{h-l-1}(w) a_{k-1}(w)}{a_{h+k}(w)}, \text{ if } l \ge 0,$$
$$\Psi_{h,k;l}(w) = \frac{w^{-l} a_{h-1}(w) a_{k+l-1}(w)}{a_{h+k}(w)}, \text{ if } l \le -1.$$

Then we put $w = z\widetilde{G}(0,0|z)/4 = v/(1+v^2)$ and we obtain that (compare with the proof of Proposition 4.2)

$$\lim_{k\to\infty} \sum_{l=-k}^{h} \widetilde{\Psi}_{h,k;l}(z) = \frac{(1+v^2)(1-v^{h+1})}{(1-v)^2},$$
$$E(z) = \frac{1+6v^2+v^4}{(1+v^2)(1-v)^2}.$$

Then

$$\widetilde{\Psi}_h(z) = \frac{(1+6v^2+v^4)(1-v^{h+1})}{(1-v)^4},$$

$$\sum_{n=0}^{\infty} \mathbb{E}[m_n^x] z^n = \frac{1+6v^2+v^4}{(1-v)^4} \sum_{h\geq 0} v^{h+1} \overset{z\to 1^-}{\sim} \frac{1}{2^{3/4}(1-z)^{5/4}},$$

whence $\mathbb{E}[m_n^x] \overset{n\to\infty}{\sim} 2^{-3/4} n^{1/4}/\Gamma(5/4)$ and we are done. □

Theorem 5.2. $\mathbb{E}[M_n^y] \overset{n\to\infty}{\sim} \sqrt{\frac{8}{\pi}}\, n^{1/2}$.

Proof. Let $m_n^y = \max\{S_i^y : 0 \leq i \leq n\}$, then $\mathbb{E}[M_n^y] = 2\mathbb{E}[m_n^y]$. Denote by $\widehat{\Psi}_h(z)$ the generating function of $\{\mathbb{P}\,(m_n^y \leq h)\}_{n\geq 0}$. Then

$$\widehat{\Psi}_h(z) = \lim_{k\to\infty} \sum_{l=-k}^{h} \widehat{\Psi}_{h,k;l}(z),$$

where $\widehat{\Psi}_{h,k;l}(z)$ is the generating function of the probabilities of the n-step paths (see Figure 2), such that $-k \leq S_i^y \leq h$ for $0 \leq i \leq n$, $S_n^y = l$. The functions $\widehat{\Psi}_{h,k;l}(z)$ are determined by the linear system (where only the non-zero terms are displayed)

$$\begin{bmatrix} \begin{bmatrix} \mathbf{A}_{h-1} \end{bmatrix} & & & & \\ & & -z/4 & & \\ & -z/2 & (1-z/2) & -z/2 & \\ & & -z/4 & & \\ & & & & \begin{bmatrix} \mathbf{A}_{k-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \widehat{\Psi}_{h,k;h}(z) \\ \vdots \\ \widehat{\Psi}_{h,k;1}(z) \\ \widehat{\Psi}_{h,k;0}(z) \\ \widehat{\Psi}_{h,k;-1}(z) \\ \vdots \\ \widehat{\Psi}_{h,k;-k}(z) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Denote by $b_{h,k}$ the determinant of the previous matrix. For $h \geq 2$ elementary computation lead to

$$\widehat{\Psi}_h(z) = \lim_{k\to\infty} \frac{1}{2b_{h,k}} \left(a_{k-1} \sum_{l=0}^{h} \left(\frac{z}{2}\right)^l a_{h-l-1} + a_{h-1} \sum_{l=0}^{k} \left(\frac{z}{2}\right)^l a_{k-l-1} \right)$$

The interest in $\widehat{\Psi}_h(z)$ originates in that

$$\sum_{n\geq 0} \mathbb{E}[m_n^y] z^n = \sum_{h\geq 0} \left(\frac{1}{1-z} - \widehat{\Psi}_h(z) \right). \tag{10}$$

By Lemma 4.5 it is not difficult to prove that for $h \geq 2$ (and v such that (5) holds)

$$\frac{1}{1-z} - \Psi_h(z) = \frac{1+v^2}{(1-v)^2} \frac{(1+v)v^{h+1}}{(2-(1-v)v^{2h+2})}.$$

Noting that $2-(1-v)v^{2h+2} \in [2v, 2]$ we get that as $z \to 1^-$ (and $v \to 1^-$)

$$\sum_{n\geq 0} \mathbb{E}[m_n^y] z^n \sim \frac{2}{(1-v)^3} \sim \frac{1}{\sqrt{2}(1-z)^{3/2}}.$$

Of course we have to prove that the terms in (10) corresponding to $h = 0, 1$ are negligible, but this follows from elementary computation. By [9, Corollary 2] we deduce that $\mathbb{E}[m_n^y] \overset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\, n^{1/2}$ and we are done. □

6 Scaling limits

In the preceding sections we have seen that the expected values of the distances (with various meanings of this word) reached in n steps are of order $n^{1/4}$ for the horizontal direction and of order $n^{1/2}$ for the vertical direction. These results lead us to a natural question: what is the asymptotic behaviour of the process where the horizontal component of the position after n steps is divided by $n^{1/4}$ and the vertical component is divided by $n^{1/2}$? Of course we have to make this question more precise.

In order to study the scaling of the process we choose a suitable realization for the sequence $\{S_n\}_{n\geq 0}$: let $X = \{X_n\}_{n\geq 0}$ be a sequence of random variables representing a simple random walk on $\mathbb{Z}$, and $Y = \{Y_n\}_{n\geq 0}$ be a sequence representing the random walk on $\mathbb{Z}$ moving according to Figure 2. Choose X and Y to be independent and let $X_0 = Y_0 \equiv 0$ a.s.. Moreover, let L_k be the number of loops performed by Y up to time k, that is

$$L_k = \sum_{i=0}^{k-1} \mathbb{1}_{Y_i=0, Y_{i+1}=0}.$$

Clearly, $S_n = (X_{L_n}, Y_n)$ is a realization of the position of the simple random walker on $\mathbf{C}_2$ at time n. We are now able to define, by linear interpolation of the three discrete time processes X, Y and L, a continuous time process $(X_{L_{nt}}, Y_{nt})$.

Theorem 6.1.

$$\left(\frac{X_{L_{nt}}}{\sqrt[4]{n}}, \frac{Y_{nt}}{\sqrt{n}}\right)_{t\geq 0} \xrightarrow{Law} \left(W_{L_t^0(B)}, B_t\right)_{t\geq 0}, \tag{11}$$

where W and B are two independent Brownian motions and $L_t^0(B)$ is the local time at 0 of B.

The theorem will be a consequence of Proposition 6.4 and Proposition 6.5. We introduce the following notion of convergence of stochastic processes (see Definition 2.2 of [5]).

Definition 6.2. *A sequence of $\mathbb{R}^k$-valued stochastic processes $(Z_t^n; t \geq 0)_{n\geq 0}$ converges to a process $(Z_t; t \geq 0)$ in probability uniformly on compact intervals if for all $t \geq 0$, as $n \to \infty$*

$$\sup_{s\leq t} \|Z_s^n - Z_s\| \xrightarrow{\mathbb{P}} 0,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^k$ (for instance $\|\cdot\|_1$). We will briefly write

$$(Z_t^n; t \geq 0) \xrightarrow{U.P.} (Z_t; t \geq 0).$$

Since U.P. convergence of a vector is equivalent to U.P. convergence of its components and implies convergence in distribution, in order to prove Theorem 6.1 it will suffice to prove that each component in (11) U.P. converges to the corresponding limit.

The main idea is that Y and L are not much different from, respectively, a simple random walk Y' on $\mathbb{Z}$ and the process L' which counts the visits of Y' to 0. There is a natural correspondence between Y and Y', so let us define Y' and some other auxiliary variables which will be needed in the sequel. Given Y, let l and N be the processes which respectively count its returns to 0 (not including the loops and counting time 0 as the first "return") and the time spent not looping at 0. Namely, let $l_0 = 1$, and $l_k = 1 + \sum_{i=0}^{k-1} 1\!\!1_{Y_{i+1}=0, Y_i \neq 0}$ and $N_k = \sum_{i=0}^{k-1} 1\!\!1_{Y_i \neq Y_{i+1}}$ for $k \geq 1$. Clearly $N_k = k - L_k$. Moreover, note that for $k \geq 1$,

$$\sum_{i=0}^{l_k - 1} \tau_i \leq L_k \leq \sum_{i=0}^{l_k} \tau_i, \tag{12}$$

where $\tau = \{\tau_i\}_{i \geq 0}$ is a suitable sequence of iid random variables with geometric distribution of parameter $1/2$. Now define a simple random walk Y' on $\mathbb{Z}$ by $Y_n = Y'_{N_n}$. Then $L'_k = \sum_{i=0}^{k} 1\!\!1_{Y'_i = 0}$ (L' counts the visits at 0 or, equivalently, the returns to 0). We note that $l_n = L'_{N_n}$ and $0 \leq l_n \leq L'_n$. We first prove a property of L.

Lemma 6.3.

$$\frac{L_n}{\sqrt{n}} \xrightarrow{Law} |\mathcal{N}(0,1)|.$$

Proof. The main ideas of the proof are the facts that $L'_n/\sqrt{n} \to |\mathcal{N}(0,1)|$ and that N_n is not much different from n. Indeed one can easily prove the first fact (for the distribution of $(L'_{2n} - 1)$ see [7, Chapter III, Exercise 10]). By (12), the thesis is a consequence of

$$\frac{\sum_{i=0}^{l_n} \tau_i}{\sqrt{n}} \xrightarrow{Law} |\mathcal{N}(0,1)|. \tag{13}$$

By the strong law of large numbers and Slutzky's theorem,

$$\frac{\sum_{i=0}^{L'_n} \tau_i}{\sqrt{n}} = \frac{\sum_{i=0}^{L'_n} \tau_i}{L'_n} \cdot \frac{L'_n}{\sqrt{n}} \xrightarrow{Law} |\mathcal{N}(0,1)|.$$

Then (13) will follow once we show that $\sum_{i=0}^{l_n} \tau_i / \sum_{i=0}^{L'_n} \tau_i \xrightarrow{\mathbb{P}} 1$. Indeed note that

$$L'_n = L'_{N_n} + R'_n = l_n + R'_n, \tag{14}$$

where R'_n is the number of visits to 0 of Y' between time N_n and time n. Let T_n be the time Y' first visits 0 after time N_n, and for any $j \geq 0$ let L''_j be the number of visits to 0 before time j of the random walk $Y''_m := Y'_{m+T_n}$. Clearly L''_j is independent of $\{Y'_k\}_{k=0}^{N_n}$ and has the same distribution of L'_j. Then

$$\frac{\sum_{i=0}^{l_n} \tau_i}{\sum_{i=0}^{L'_n} \tau_i} = 1 - \frac{\sum_{i=l_n+1}^{L'_n} \tau_i}{\sum_{i=0}^{L'_n} \tau_i} = 1 - \frac{\sum_{i=1}^{R'_n} \widetilde{\tau}_i}{\sum_{i=0}^{L'_n} \tau_i},$$

where $\widetilde{\tau}_i = \tau_{i+l_n}$ and $\sum_{i=1}^{R'_n} \widetilde{\tau}_i$ is equal to zero if $L'_n = l_n$. We are left with the proof that $\sum_{i=1}^{R'_n} \widetilde{\tau}_i / \sum_{i=0}^{L'_n} \tau_i \xrightarrow{\mathbb{P}} 0$. Since $0 \leq R'_n \leq L''_{n-N_n}$, it suffices to prove that $\sum_{i=1}^{L''_{n-N_n}} \widetilde{\tau}_i / \sum_{i=0}^{L'_n} \tau_i \xrightarrow{\mathbb{P}}$

0. Fix $\varepsilon > 0$: since $\sum_{i=0}^{L'_n} \tau_i \geq n - N_n$ we have that

$$\mathbb{P}\left(\frac{\sum_{i=1}^{L''_{n-N_n}} \widetilde{\tau}_i}{\sum_{i=0}^{L'_n} \tau_i} > \varepsilon\right) \leq \sum_{k=1}^{n} \mathbb{P}\left(\frac{\sum_{i=1}^{L''_{n-N_n}} \widetilde{\tau}_i}{n - N_n} > \varepsilon,\ n - N_n = k\right)$$
$$= \sum_{k=1}^{n} \mathbb{P}\left(\frac{\sum_{i=1}^{L''_k} \widetilde{\tau}_i}{k} > \varepsilon\right) \mathbb{P}(n - N_n = k).$$

Now fix $\delta > 0$ and choose $M = M(\delta)$ such that $\mathbb{P}(\sum_{i=1}^{L''_k} \widetilde{\tau}_i/k > \varepsilon) < \delta$ for all $k \geq M$ (this is possible by the law of large numbers using the facts that L'' and $\widetilde{\tau}$ are independent and $L''_k/k \xrightarrow{\mathbb{P}} 0$). Then

$$\sum_{k=1}^{n} \mathbb{P}\left(\frac{\sum_{i=1}^{L''_k} \widetilde{\tau}_i}{k} > \varepsilon\right) \mathbb{P}(n - N_n = k) \leq \mathbb{P}(n - N_n < M) + \delta.$$

Since $n - N_n \xrightarrow{\mathbb{P}} \infty$ we are done.

□

Proposition 6.4.
$$\left(\frac{1}{\sqrt{n}} Y_{nt}, \frac{1}{\sqrt{n}} L_{nt}\right)_{t\geq 0} \xrightarrow{U.P.} \left(B_t, L_t^0(B)\right)_{t\geq 0}.$$

Proof. Consider the processes Y' and L' defined before Lemma 6.3 and by interpolation define the sequence of two-dimensional continuous time processes $\left((\frac{1}{\sqrt{n}} Y'_{nt}, \frac{1}{\sqrt{n}} L'_{nt}); t \geq 0\right)_{n\geq 0}$. Then by Theorem 3.1 of [5] we have that

$$\left(\frac{1}{\sqrt{n}} Y'_{nt}, \frac{1}{\sqrt{n}} L'_{nt}\right)_{t\geq 0} \xrightarrow{U.P.} \left(B_t, L_t^0(B)\right)_{t\geq 0}.$$

To prove our statement, it suffices to show that these two properties hold:

$$(A) : \left(\frac{1}{\sqrt{n}} \left(Y_{\lfloor nt \rfloor} - Y'_{\lfloor nt \rfloor}\right)\right)_{t\geq 0} \xrightarrow{U.P.} 0$$
$$(B) : \left(\frac{1}{\sqrt{n}} \left(L_{\lfloor nt \rfloor} - L'_{\lfloor nt \rfloor}\right)\right)_{t\geq 0} \xrightarrow{U.P.} 0.$$

Note that Y' is the sum of iid increments ξ_i such that $\mathbb{P}(\xi_i = \pm 1) = 1/2$, hence

$$Y_{\lfloor nt \rfloor} = Y'_{\lfloor nt \rfloor - L_{\lfloor nt \rfloor}} = \sum_{i=1}^{\lfloor nt \rfloor - L_{\lfloor nt \rfloor}} \xi_i$$
$$= \sum_{i=1}^{\lfloor nt \rfloor} \xi_i - \sum_{i=\lfloor nt \rfloor - L_{\lfloor nt \rfloor}+1}^{\lfloor nt \rfloor} \xi_i = Y'_{\lfloor nt \rfloor} - \sum_{i=1}^{L_{\lfloor nt \rfloor}} \tilde{\xi}_i,$$

where $\tilde{\xi}_i = \xi_{\lfloor nt\rfloor - L_{\lfloor nt\rfloor}+i}$ (and $\sum_{i=1}^{L_{\lfloor nt\rfloor}} \tilde{\xi}_i = 0$ if $L_{\lfloor nt\rfloor} = 0$). Thus $Y_{\lfloor nt\rfloor} - Y'_{\lfloor nt\rfloor} = \sum_{i=1}^{L_{\lfloor nt\rfloor}} \tilde{\xi}_i$. Then (A) follows from

$$\sup_{s\le t}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{L_{\lfloor ns\rfloor}} \tilde{\xi}_i\right| \xrightarrow{\mathbb{P}} 0. \tag{15}$$

Indeed

$$\sup_{s\le t}\left|\sum_{i=1}^{L_{\lfloor ns\rfloor}} \tilde{\xi}_i\right| \le \max_{k\le L_{\lfloor nt\rfloor}}\left|\sum_{i=1}^{k} \tilde{\xi}_i\right|,$$

and if we denote by $M_n = \max_{k\le n}\sum_{i=1}^k \tilde{\xi}_i$ and by $m_n = \min_{k\le n}\sum_{i=1}^k \tilde{\xi}_i$, clearly M_n and $-m_n$ are identically distributed, and $\max_{k\le L_{\lfloor nt\rfloor}}\left|\sum_{i=1}^k \tilde{\xi}_i\right| = \max\left\{M_{L_{\lfloor nt\rfloor}}, -m_{L_{\lfloor nt\rfloor}}\right\}$ Hence to prove (15) it suffices to show that

$$\frac{1}{\sqrt{n}} M_{L_{\lfloor nt\rfloor}} \xrightarrow{\mathbb{P}} 0.$$

The distribution of M_n is well known (see [7, Chapter III.7]), and it is easy to show that $M_n/\sqrt{n} \overset{Law}{\to} |\mathcal{N}(0,1)|$. Noting that $L_{\lfloor nt\rfloor}$ is independent of M_k, we have that

$$\begin{aligned}
\mathbb{P}(M_{L_{\lfloor nt\rfloor}} > \varepsilon\sqrt{n}) &= \sum_{k=0}^{\lfloor nt\rfloor} \mathbb{P}(M_k > \varepsilon\sqrt{n})\mathbb{P}(L_{\lfloor nt\rfloor} = k)\\
&\le \sum_{\varepsilon\sqrt{n}<k\le\alpha\sqrt{\lfloor nt\rfloor}} \mathbb{P}(M_k > \varepsilon\sqrt{n})\mathbb{P}(L_{\lfloor nt\rfloor} = k) + \sum_{k>\alpha\sqrt{\lfloor nt\rfloor}} \mathbb{P}(L_{\lfloor nt\rfloor} = k)\\
&\le \mathbb{P}(M_{\lfloor\alpha\sqrt{\lfloor nt\rfloor}\rfloor} > \varepsilon\sqrt{n}) + \mathbb{P}(L_{\lfloor nt\rfloor} > \alpha\sqrt{\lfloor nt\rfloor}).
\end{aligned}$$

By Lemma 6.3, for any positive ε' and t there exist α and n' such that $\mathbb{P}(L_{\lfloor nt\rfloor} > \alpha\sqrt{\lfloor nt\rfloor}) < \varepsilon'$ for all $n \ge n'$. On the other hand for any given ε, α

$$\mathbb{P}\left(\frac{M_{\lfloor\alpha\sqrt{\lfloor nt\rfloor}\rfloor}}{\sqrt{\alpha}\sqrt[4]{\lfloor nt\rfloor}} > \frac{\varepsilon\sqrt[4]{n}}{\sqrt{\alpha}\sqrt[4]{t}}\right) \overset{n\to\infty}{\longrightarrow} 0,$$

whence (A) is proven.

Now, let us address to (B). We first note that we may consider a mapping between the number of steps taken by Y' and the ones taken by Y. Indeed when Y' has taken $\lfloor nt\rfloor$ steps, then Y has taken $\lfloor nt\rfloor + \sum_{i\le L'_{\lfloor nt\rfloor}} \tau_i$ steps (that is, if $Y'_{\lfloor nt\rfloor} = 0$ we decide to count for Y all the loops it performs after this last return to 0). Let us write

$$\begin{aligned}
\frac{1}{\sqrt{n}}\left(L'_{\lfloor nt\rfloor} - L_{\lfloor nt\rfloor}\right) = \frac{1}{\sqrt{n}}&\left(L'_{\lfloor nt\rfloor} - L_{\lfloor nt\rfloor+\sum_{i\le L'_{\lfloor nt\rfloor}}\tau_i}\right) +\\
&\frac{1}{\sqrt{n}}\left(L_{\lfloor nt\rfloor+\sum_{i\le L'_{\lfloor nt\rfloor}}\tau_i} - L_{\lfloor nt\rfloor}\right) =: I + II.
\end{aligned} \tag{16}$$

We prove that $I \stackrel{U.P.}{\to} 0$. Indeed $L_{\lfloor nt\rfloor + \sum_{i\le L'_{\lfloor nt\rfloor}}\tau_i} = \sum_{i\le L'_{\lfloor nt\rfloor}}\tau_i$, thus

$$\sup_{s\le t}\left|L'_{\lfloor ns\rfloor} - \sum_{i\le L'_{\lfloor ns\rfloor}}\tau_i\right| = \sup_{s\le t}\left|\sum_{i\le L'_{\lfloor ns\rfloor}}(1-\tau_i)\right| \le \max_{k\le L'_{\lfloor nt\rfloor}}\left|\sum_{i\le k}(1-\tau_i)\right|.$$

Now choose $\delta > 0$. By independence of L' and τ we have that the probability that $\max_{k\le L'_{\lfloor nt\rfloor}}\left|\sum_{i\le k}(1-\tau_i)\right|$ is larger than $\varepsilon\sqrt{n}$ is bounded by

$$\mathbb{P}\left(L'_{\lfloor nt\rfloor} > \alpha\sqrt{\lfloor nt\rfloor}\right) + \sum_{j\le\alpha\sqrt{\lfloor nt\rfloor}}\mathbb{P}\left(\max_{k\le j}\left|\sum_{i\le k}(1-\tau_i)\right| > \varepsilon\sqrt{n}\right)\mathbb{P}\left(L'_{\lfloor nt\rfloor} = j\right).$$

The first term is smaller than δ if α and n are sufficiently large. As for the second term, it is clearly less or equal to

$$\mathbb{P}\left(\max_{k\le\alpha\sqrt{\lfloor nt\rfloor}}\left|\sum_{i\le k}(1-\tau_i)\right| > \varepsilon\sqrt{n}\right). \tag{17}$$

Observe that, by the law of large numbers, for any positive ε' and δ there exists $k_0 = k_0(\varepsilon',\delta)$ such that for all $k \ge k_0$

$$\mathbb{P}\left(\left|\frac{\sum_{i\le k}(1-\tau_i)}{k}\right| < \varepsilon'\right) \ge 1-\delta.$$

Hence (17) is less or equal to

$$\mathbb{P}\left(\max_{k\le k_0}\left|\sum_{i\le k}(1-\tau_i)\right| > \varepsilon\sqrt{n}\right) + \mathbb{P}\left(\max_{k_0\le k\le\alpha\sqrt{\lfloor nt\rfloor}}\left|\sum_{i\le k}(1-\tau_i)\right| > \varepsilon\sqrt{n}\right).$$

The first term clearly tends to 0 as n grows to infinity, while the second term is not larger than

$$\delta + \mathbb{P}\left(\max_{k_0\le k\le\alpha\sqrt{\lfloor nt\rfloor}}\left|\sum_{i\le k}(1-\tau_i)\right| > \varepsilon\sqrt{n}, \left|\frac{\sum_{i\le k}(1-\tau_i)}{k}\right| < \varepsilon' \quad \forall k\ge k_0\right).$$

But if $|\sum_{i\le k}(1-\tau_i)|/k < \varepsilon'$ for all $k \ge k_0$, then

$$\sup_{k_0\le k\le\alpha\sqrt{\lfloor nt\rfloor}}\left|\sum_{i\le k}(1-\tau_i)\right| < \alpha\varepsilon'\sqrt{\lfloor nt\rfloor},$$

which is smaller than $\varepsilon\sqrt{n}$ if ε' is sufficiently small. This proves that $I \stackrel{U.P.}{\to} 0$.

We now prove that $II \stackrel{U.P.}{\to} 0$. Indeed

$$0 \le L_{\lfloor ns\rfloor + \sum_{i\le L'_{\lfloor ns\rfloor}}\tau_i} - L_{\lfloor ns\rfloor} \le \sum_{i\le L'_{\lfloor ns\rfloor}}\tau_i - \sum_{i\le l_{\lfloor ns\rfloor}-1}\tau_i = \sum_{i=l_{\lfloor ns\rfloor}}^{L'_{\lfloor ns\rfloor}}\tau_i.$$

Using (14) and the definitions of L'' and $\widetilde{\tau}$ thereafter, we have that $\sum_{i=l_{\lfloor ns\rfloor}}^{L'_{\lfloor ns\rfloor}} \tau_i \le \sum_{i=0}^{L''_{\lfloor ns\rfloor - N_{\lfloor ns\rfloor}}} \widetilde{\tau}_i$, and for any positive ε',

$$\sup_{s\le t} \sum_{i=0}^{L''_{\lfloor ns\rfloor - N_{\lfloor ns\rfloor}}} \tilde{\tau}_i = \max\left\{ \sup_{s\le\varepsilon'} \sum_{i=0}^{L''_{\lfloor ns\rfloor - N_{\lfloor ns\rfloor}}} \tilde{\tau}_i,\ \sup_{\varepsilon'<s\le t} \sum_{i=0}^{L''_{\lfloor ns\rfloor - N_{\lfloor ns\rfloor}}} \tilde{\tau}_i \right\} =: \max(C, D).$$

Choose $\delta > 0$. Let us show that $\mathbb{P}(C > \varepsilon\sqrt{n}) < \delta$ if ε' is sufficiently small and n sufficiently large. Indeed

$$\sup_{s\le\varepsilon'} \sum_{i=0}^{L''_{\lfloor ns\rfloor - N_{\lfloor ns\rfloor}}} \tilde{\tau}_i \le \sum_{i=0}^{L''_{\lfloor n\varepsilon'\rfloor}} \tilde{\tau}_i,$$

and by independence of $L''_{\lfloor n\varepsilon'\rfloor}$ and $\tilde{\tau}$,

$$\mathbb{P}\left(\sum_{i\le L''_{\lfloor n\varepsilon'\rfloor}} \tilde{\tau}_i > \varepsilon\sqrt{n} \right) \le \mathbb{P}\left(L''_{\lfloor n\varepsilon'\rfloor} > \alpha\sqrt{\lfloor n\varepsilon'\rfloor} \right) + \mathbb{P}\left(\sum_{i\le\alpha\sqrt{\lfloor n\varepsilon'\rfloor}} \tilde{\tau}_i > \varepsilon\sqrt{n} \right). \tag{18}$$

Choose α such that $\mathbb{P}\left(L''_{\lfloor n\varepsilon'\rfloor} > \alpha\sqrt{\lfloor n\varepsilon'\rfloor} \right) < \delta$ for n sufficiently large, and then ε' sufficiently small such that $\mathbb{P}\left(\sum_{i\le\alpha\sqrt{\lfloor n\varepsilon'\rfloor}} \widetilde{\tau}_i > \varepsilon\sqrt{n} \right) < \delta$.

Now keep ε' fixed. In order to prove that $\mathbb{P}(D > \varepsilon\sqrt{n}) < 3\delta$ for n sufficiently large, we observe that for any positive ε'' there exists $\bar{n}$ such that for all $n \ge \bar{n}$, $\mathbb{P}\left(N_n \ge n(1-\varepsilon'') \right)$ is larger than $1-\delta$. Then for all n such that $\lfloor n\varepsilon'\rfloor \ge \bar{n}$, $\sup_{\varepsilon'<s\le t}\left(L''_{\lfloor ns\rfloor - N_{\lfloor ns\rfloor}} \right) \le L''_{\lfloor nt\varepsilon''\rfloor}$ with probability larger than $1-\delta$. Hence, as in (18) we have that, for α and n sufficiently large and ε'' sufficiently small.

$$\begin{aligned} \mathbb{P}\left(D > \varepsilon\sqrt{n} \right) &\le \delta + \mathbb{P}\left(\sum_{i\le L''_{\lfloor nt\varepsilon''\rfloor}} \tilde{\tau}_i > \varepsilon\sqrt{n} \right) \\ &\le \delta + \mathbb{P}\left(L''_{\lfloor nt\varepsilon''\rfloor} > \alpha\sqrt{\lfloor nt\varepsilon''\rfloor} \right) + \mathbb{P}\left(\sum_{i\le\alpha\sqrt{\lfloor nt\varepsilon''\rfloor}} \tilde{\tau}_i > \varepsilon\sqrt{n} \right) < 3\delta. \end{aligned}$$

□

Proposition 6.5.

$$\left(\frac{X_{L_{nt}}}{\sqrt[4]{n}} \right)_{t\ge0} \xrightarrow{U.P.} \left(W_{L_t^0(B)} \right)_{t\ge0}.$$

Proof. Clearly the statement may be rephrased as

$$\left(\frac{X_{L_{n^2t}}}{\sqrt{n}} \right)_{t\ge0} \xrightarrow{U.P.} \left(W_{L_t^0(B)} \right)_{t\ge0}.$$

Note that (writing L_s^0 instead of $L_s^0(B)$)

$$\sup_{s\le t}\left|\frac{X_{L_{n^2s}}}{\sqrt{n}}-W_{L_s^0}\right|\le\sup_{s\le t}\left|\frac{X_{L_{n^2s}}-X_{nL_s^0}}{\sqrt{n}}\right|+\sup_{s\le t}\left|\frac{X_{nL_s^0}}{\sqrt{n}}-W_{L_s^0}\right|=(A)+(B).$$

We have to show that (A) and $(B)\overset{U.P.}{\to}0$: it suffices to prove the statement for $L_{\lfloor n^2s\rfloor}$ and $\lfloor nL_s^0\rfloor$ instead of L_{n^2s} and nL_s^0 respectively. Represent $X_k=\sum_1^k\widetilde{\eta}_i$, where $\widetilde{\eta}=\{\widetilde{\eta}_i\}_{i\ge1}$ is an iid family such that $\mathbb{P}(\widetilde{\eta}_i=\pm1)=1/2$, then

$$\sup_{s\le t}\left|X_{L_{\lfloor n^2s\rfloor}}-X_{\lfloor nL_s^0\rfloor}\right|=\sup_{s\le t}\left|\sum_{i=1}^{n\Delta_{n,s}}\eta_i\right|,$$

where $n\Delta_{n,s}:=|L_{\lfloor n^2s\rfloor}-\lfloor nL_s^0\rfloor|$ and $\eta_i=\widetilde{\eta}_{\min\{L_{\lfloor n^2s\rfloor},\lfloor nL_s^0\rfloor\}+i}$. Note that $(\Delta_{n,t})_{t\ge0}\overset{U.P.}{\to}0$, indeed

$$\Delta_{n,t}\le\left|\frac{L_{\lfloor n^2t\rfloor}}{n}-L_t^0\right|+\left|L_t^0-\frac{\lfloor nL_t^0\rfloor}{n}\right|\le\left|\frac{L_{\lfloor n^2t\rfloor}}{n}-L_t^0\right|+\frac{1}{n},$$

and both these summands tends U.P. to 0 (the first by Proposition 6.4). In order to show that

$$\mathbb{P}\left(\sup_{s\le t}\left|\sum_{i=1}^{n\Delta_{n,s}}\eta_i\right|>\varepsilon\sqrt{n}\right)\overset{n\to\infty}{\to}0$$

it suffices to prove that

$$\mathbb{P}\left(\max_{k\le\sup_{s\le t}(n\Delta_{n,s})}\sum_{i=1}^{k}\eta_i>\varepsilon\sqrt{n}\right)\overset{n\to\infty}{\to}0.$$

Now put $M_n=\max_{k\le n}\sum_{i=1}^k\eta_i$ and recall that $(n\Delta_{n,s})_{n\ge0,s\le t}$ and η are independent: this last probability may be written as

$$\begin{aligned}&\sum_{j=0}^{\infty}\mathbb{P}(M_j>\varepsilon\sqrt{n})\mathbb{P}\left(\sup_{s\le t}(n\Delta_{n,s})=j\right)\\&\le\mathbb{P}\left(\sup_{s\le t}(\Delta_{n,s})>\alpha\right)+\mathbb{P}(M_{\lfloor n\alpha\rfloor}>\varepsilon\sqrt{n}).\end{aligned}$$

For all positive α, $\mathbb{P}\left(\sup_{s\le t}(\Delta_{n,s})>\alpha\right)$ can be made arbitrarily small if n is picked large enough, while

$$\mathbb{P}\left(M_{\lfloor n\alpha\rfloor}>\varepsilon\sqrt{n}\right)\overset{n\to\infty}{\to}\mathbb{P}\left(|\mathcal{N}(0,1)|>\frac{\varepsilon}{\sqrt{\alpha}}\right).$$

Thus if $\delta>0$ is fixed, by choosing α sufficiently small we get $\mathbb{P}\left(M_{\lfloor n\alpha\rfloor}>\varepsilon\sqrt{n}\right)<\delta$ if n is sufficiently large. This proves $(A)\overset{U.P.}{\to}0$.

Now note that X_n and L_s^0 are independent, so we may think of $X_{\lfloor nL_s^0\rfloor}$ as defined on a product probability space $\Omega_1\otimes\Omega_2$ (that is $X_{\lfloor nL_s^0\rfloor}(\omega_1,\omega_2)=X_{\lfloor nL_s^0(\omega_2)\rfloor}(\omega_1)$). For any fixed ω_2 we have that

$$\mathbb{P}_1\left(\sup_{s\le t}\left|\frac{X_{\lfloor nL_s^0(\omega_2)\rfloor}}{\sqrt{n}}-W_{L_s^0(\omega_2)}\right|>\varepsilon\right)\overset{n\to\infty}{\to}0.\tag{19}$$

Indeed it is known (see [5]) that $\mathbb{P}_1\left(\sup_{s\leq t}\left|\frac{X_{ns}}{\sqrt{n}}-W_s\right|>\varepsilon\right)\overset{n\to\infty}{\to}0$, and since for any fixed ω_2, as $s\uparrow t$, $L^0_s(\omega_2)\uparrow L^0_t(\omega_2)$,

$$\sup_{s\leq t}\left|\frac{X_{\lfloor nL^0_s(\omega_2)\rfloor}}{\sqrt{n}}-W_{L^0_s(\omega_2)}\right|\leq\sup_{s\leq L^0_t(\omega_2)}\left|\frac{X_{\lfloor ns\rfloor}}{\sqrt{n}}-W_s\right|,$$

whence (19). Thus, putting

$$A_n:=\left\{(\omega_1,\omega_2):\sup_{s\leq t}\left|\frac{X_{\lfloor nL^0_s(\omega_2)\rfloor}(\omega_1)}{\sqrt{n}}-W_{L^0_s(\omega_2)}(\omega_1)\right|>\varepsilon\right\},$$

we have that $\mathbb{P}_1(A_n)\to 0$, by Fubini and the dominated convergence theorem we get $\mathbb{P}_1\otimes\mathbb{P}_2(A_n)\to 0$ and we are done. $\square$

7 Final remarks

The results of the previous sections show that the random walker on $\mathbf{C}_2$ spends most of the time walking along the vertical direction. One would ask to what extent the resemblance between the simple random walk on $\mathbb{Z}$ and the vertical component of the simple random walk on $\mathbf{C}_2$ is apparent. The answer is that, to leading order, the expected values of the distances after n steps of these two walks are indistinguishable. Indeed if we denote by X_n the position at time n of the walker on $\mathbb{Z}$, one could easily compute

$$\mathbb{E}[|X_n|]\overset{n\to\infty}{\sim}\sqrt{\frac{\pi}{2}}\,n^{1/2},$$

which coincides with the estimate of Theorem 3.2. Moreover, a comparison between Theorem 4.10 and [15, Theorem 2.14] and between Theorem 5.2 and [15, Theorem 3.4] shows the same coincidence for the estimates of the maximal deviation and the maximal span respectively.

The inhomogeneity of $\mathbf{C}_2$ results in the difference between the behaviour of the horizontal and vertical components of the walk. Indeed while for any positive integer d the expected value of the distance from the origin of the simple random walker on $\mathbb{Z}^d$, after n steps, is of order $n^{1/2}$ (this may be easily proven via a conditioning argument, with respect to the proportions of time spent along each of the d main directions), the expected value of the horizontal distance on $\mathbf{C}_2$ is of order $n^{1/4}$, that is on this direction we have a subdiffusive behaviour. Of course this is due to the delay observed on the x-axis while the walker explores the teeth of the comb (recall that the simple random walk on $\mathbb{Z}$, although recurrent, is zero-recurrent, that is the expected value of the first return time to the origin is infinite).

The difference between the two projections of the walk is remarkable also when one properly rescales the process as we did in Section 6. Indeed the space where the "rescaled walker" lives is $\mathbb{R}^2$, endowed with the topology for which any path between two points (x_1,y_1) and (x_2,y_2) must necessarily include the three edges (x_1,y_1)–$(x_1,0)$, $(x_1,0)$–$(x_2,0)$ and $(x_2,0)$–(x_2,y_2). This reflects on the limiting process: the horizontal component may change only when the vertical one passes through 0. In fact, as we proved in Theorem 6.1, $S^y_{nt}/n^{1/2}$ converges to a standard Brownian motion B, (the walk on the vertical direction is, to leading

order, unaffected by the bias of the random holding time at zero), while $S^x_{nt}/n^{1/4}$ converges to a Brownian motion whose clock is the local time at zero of B.

The fact that the walker on $\mathbf{C}_2$ essentially behaves like the walker on $\mathbb{Z}$ makes it clear that $\delta_w(\mathbf{C}_2)$ must be 2, as we proved by combinatorial methods in Section 4 (note that $\delta_w(\mathbb{Z}^d) = 2$ for all integer $d \geq 1$). Indeed it is very likely that the walker will exit the ball of radius n moving along some tooth of $\mathbf{C}_2$. This disproves the Einstein relation between $\delta_s(\mathbf{C}_2) = 3/2$, $\delta_f(\mathbf{C}_2) = 2$ and $\delta_w(\mathbf{C}_2) = 2$. The failure of this relation in this case is due to the inhomogeneity of this particular graph. Indeed for strongly inhomogeneous graphs the growth exponent does not give an accurate description of the "fractal properties" of the graph (the assignment of the same δ_f of $\mathbb{Z}^2$ to $\mathbf{C}_2$ disregards the topology of the two structures). It seems that defining δ_f as the growth exponent of the graph makes sense only for homogeneous graphs (like $\mathbb{Z}^d$) or self-similar graphs, which have a clear fractal nature (like the Sierpiński graph). It is our opinion that the study of the sense in which a graph has a fractal nature and what is the proper definition of its fractal dimension should require further investigations.

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